Cascades and Spectra of 2D Cahn-Hilliard-Navier-Stokes (CHNS) Turbulence

Xiang Fan[1], P H Diamond[1], Luis Chacon[2]

[1] University of California, San Diego
[2] Los Alamos National Laboratory

AGAT 2016

This research was supported by the U.S. Department of Energy, Office of Science, Office of Fusion Energy Sciences, under Award Number DE-FG02-04ER54738 and CMTFO.

Overview

- The Cahn-Hilliard Navier-Stokes (CHNS) model for spinodal decomposition in 2D symmetric binary liquid mixture resembles but is not identical to 2D Magnetohydrodynamics (MHD).
- Since 2D MHD turbulence has been well studied, it provides us with potential insight and guidance for exploring the physics of 2D CHNS turbulence.
- We compare and contrast the _cascades and spectra_ of the two systems.
- By direct numerical simulation, we find that in the elastic range, the mean square concentration spectrum H_k^{ψ} of the 2D CHNS system exhibits the same power law (−7/3) as the mean square magnetic potential spectrum H_k^A in the inverse cascade regime of 2D MHD.
- The kinetic energy spectrum of the 2D CHNS system is $E_k^K \sim k^{-3}$ if forced at large scale, suggestive of the direct enstrophy cascade power law of 2D Navier-Stokes (NS) turbulence. It is different from 2D MHD, and the difference could be explained by the difference of interface packing fraction.

UC San Diego

Outline

- What is Spinodal Decomposition
- 2D CHNS turbulence compared to 2D MHD turbulence
  - Basic equations
  - Linear elastic wave
  - Ideal quadratic conserved quantities
  - Cascades
- Important length scales and ranges of 2D CHNS turbulence
- Simulation Results
  - Simulation Setup
  - Benchmark
  - Spectral Fluxes
  - H_k^{ψ}/H_k^{A} spectrum power law
  - Energy spectrum power law
- Conclusion & discussion

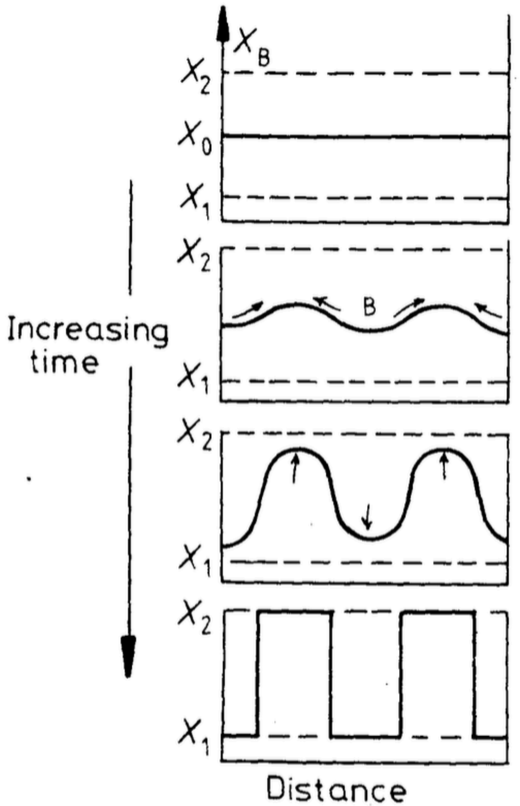
What Is Spinodal Decomposition

- The Cahn-Hilliard Navier-Stokes (CHNS) model is the standard model for binary liquid mixture undergoing **spinodal decomposition**.
- Spinodal decomposition: a 2^{nd} order phase transition for binary fluid mixture.
- Miscible phase -> Immiscible phase
- The order parameter: the local relative concentration field:

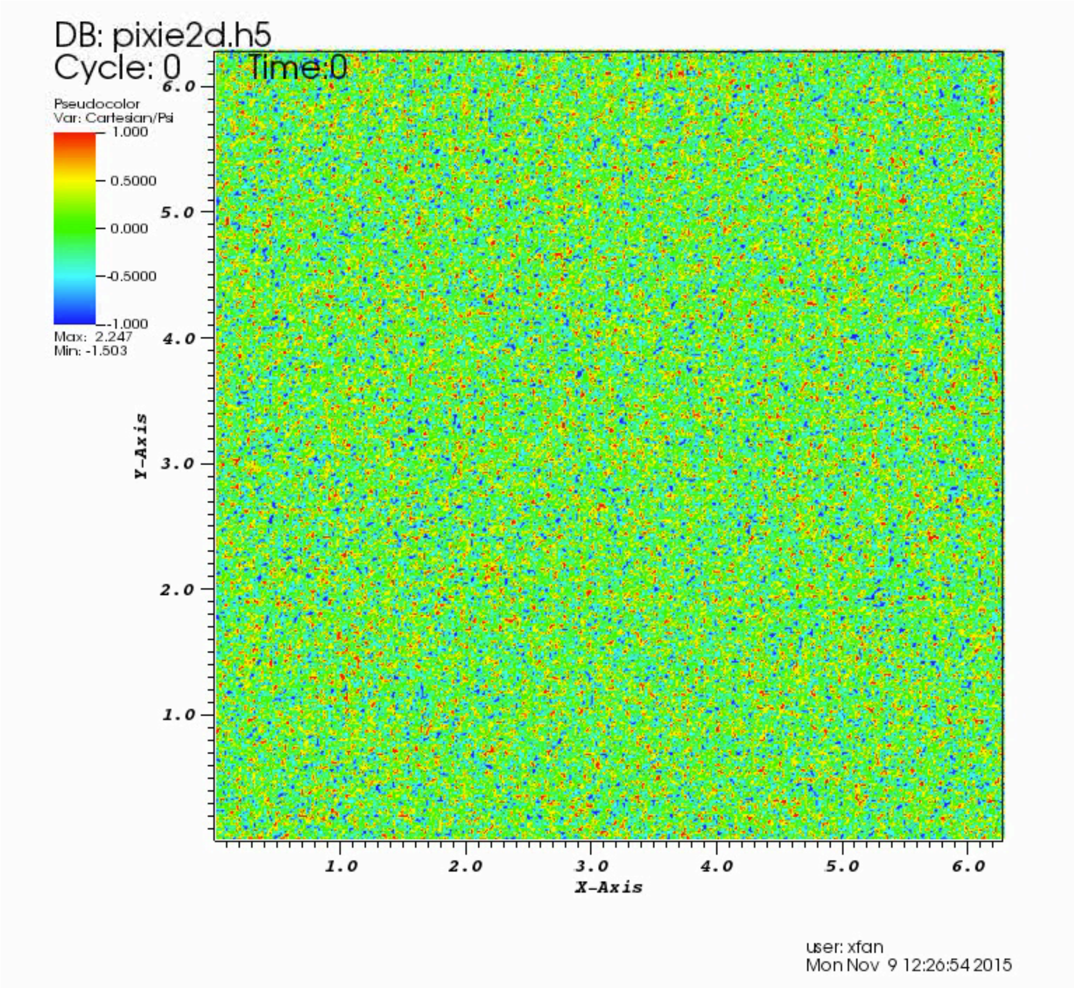
$$\psi(\vec{r},t) \stackrel{\text{def}}{=} [\rho_A(\vec{r},t) - \rho_B(\vec{r},t)]/\rho$$

- $\psi = -1$ means A-rich, $\psi = 1$ means B-rich.

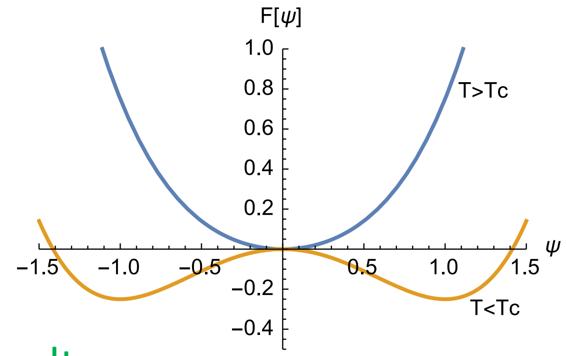
What Is Spinodal Decomposition



[Porter 1981]



What Is Spinodal Decomposition



- Landau Theory: the free energy functional:

$$\mathrm{F}(\psi) = \int d\vec{r}\Big(\underbrace{-\frac{1}{2}\psi^2 + \frac{1}{4}\psi^4}_{\text{Phase Transition}} + \underbrace{\frac{\xi^2}{2}|\nabla\psi|^2}_{\text{Curvature Penalty}}\Big)$$

Phase Transition   Curvature Penalty

- The equations for spinodal decomposition are Cahn-Hilliard Navier-Stokes (CNHS) Equations:

$$\partial_t\psi + \vec{v}\cdot\nabla\psi = D\nabla^2(-\psi + \psi^3 - \xi^2\nabla^2\psi)$$

$$\partial_t\omega + \vec{v}\cdot\nabla\omega = \frac{\xi^2}{\rho}\vec{B}_\psi\cdot\nabla\nabla^2\psi + \nu\nabla^2\omega$$

With $\vec{v}=\hat{\vec{z}}\times\nabla\phi$, $\omega = \nabla^2\phi$, $\vec{B}_\psi = \hat{\vec{z}}\times\nabla\psi$, $j_\psi = \xi^2\nabla^2\psi$.

Comparison & Contrast: Basic Equations

- The 2D CHNS Equations:

$$\partial_t \psi + \vec{v}\cdot\nabla\psi = D\nabla^2(-\psi + \psi^3 - \xi^2\nabla^2\psi)$$
$$\partial_t \omega + \vec{v}\cdot\nabla\omega = \frac{\xi^2}{\rho}\vec{B}_\psi\cdot\nabla\nabla^2\psi + \nu\nabla^2\omega$$

$-\psi$: Negative diffusion term
ψ^3: Self nonlinear term
$-\xi^2\nabla^2\psi$: Hyper-diffusion term

With $\vec{v}=\hat{z}\times\nabla\phi$, $\omega = \nabla^2\phi$, $\vec{B}_\psi = \hat{z}\times\nabla\psi$, $j_\psi = \xi^2\nabla^2\psi$

- The 2D MHD Equations:

$$\partial_t A + \vec{v}\cdot\nabla A = \eta\nabla^2 A$$
$$\partial_t \omega + \vec{v}\cdot\nabla\omega = \frac{1}{\mu_0\rho}\vec{B}\cdot\nabla\nabla^2 A + \nu\nabla^2\omega$$

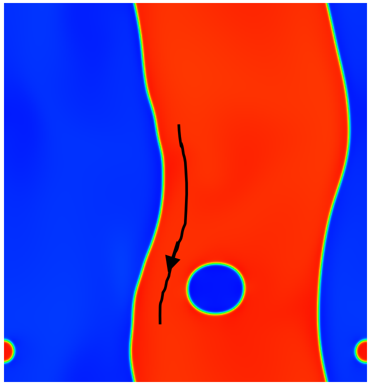
A: Simple diffusion term

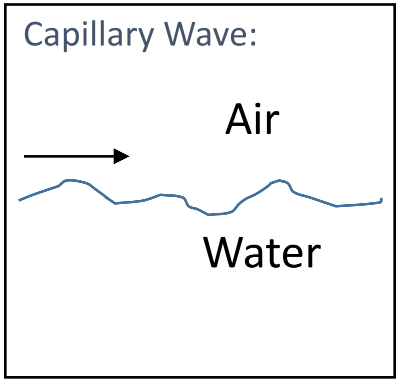
With $\vec{v}=\hat{z}\times\nabla\phi$, $\omega = \nabla^2\phi$, $\vec{B} = \hat{z}\times\nabla A$, $j = \frac{1}{\mu_0}\nabla^2 A$

- The force on fluid: $\frac{\xi^2}{\rho}\vec{B}_\psi\cdot\nabla\nabla^2\psi \Leftrightarrow \frac{1}{\mu_0\rho}\vec{B}\cdot\nabla\nabla^2 A$
- Note that the magnetic potential A is a scalar in 2D.

	2D MHD	2D CHNS
Magnetic Potential	A	ψ
Magnetic Field	$\mathbf{B}$	$\mathbf{B}_\psi$
Current	j	j_ψ
Diffusivity	η	D
Interaction strength	$\frac{1}{\mu_0}$	ξ^2

Comparison & Contrast: Linear Elastic Wave





- Alfven wave in 2D MHD:

$$\omega(k) = \pm\sqrt{\frac{1}{\mu_0\rho}}\left|\vec{k}\times\vec{B}_0\right| - \frac{1}{2}i(\eta+\nu)k^2$$

- Linear elastic wave in 2D CHNS:

$$\omega(k) = \pm\sqrt{\frac{\xi^2}{\rho}}\left|\vec{k}\times\vec{B}_{\psi 0}\right| - \frac{1}{2}i(CD+\nu)k^2$$

Where $C \equiv [-1 - 6\psi_0\nabla^2\psi_0/k^2 - 6(\nabla\psi_0)^2/k^2 - 6\psi_0\nabla\psi_0 \cdot i\mathbf{k}/k^2 + 3\psi_0^2 + \xi^2k^2]$

- The linear elastic wave in 2D CHNS is like a capillary wave: it only propagates along the boundary of the two fluids, where the gradient of concentration $B_\psi \neq 0$. Surface tension generates restoring force.
- The wave is similar to Alfven wave: they have similar dispersion relation; they both propagates along B field lines; both magnetic field and surface tension act like an elastic restoring force.
- Important difference: $\vec{B}$ fills the whole space; $\vec{B}_\psi$ is large only in the interface regions.

UC San Diego

Comparison & Contrast: Ideal quadratic conserved quantities

- **2D MHD**

1. Energy

$$E = E^K + E^B = \int (\frac{v^2}{2} + \frac{B^2}{2\mu_0}) d^2x$$

2. Mean Square Magnetic Potential

$$H^A = \int A^2 \, d^2x$$

3. Cross Helicity

$$H^C = \int \vec{v} \cdot \vec{B} d^2x$$

- **2D CHNS**

1. Energy

$$E = E^K + E^B = \int (\frac{v^2}{2} + \frac{\xi^2 B_\psi^2}{2}) d^2x$$

2. Mean Square Concentration

$$H^\psi = \int \psi^2 \, d^2x$$

3. Cross Helicity

$$H^C = \int \vec{v} \cdot \vec{B}_\psi \, d^2x$$

- “Ideal” here means $D, \eta = 0; \nu = 0.$

Comparison & Contrast: Cascades

- Turbulence cascade directions are suggested by the absolute equilibrium distributions.
- The peak of the absolute equilibrium distribution for each quadratic conserved quantity is a good indicator of the corresponding cascade direction.
- The spectrum is peaked at high k -> excitation relaxes towards high k -> direct cascade.
- The spectrum is peaked at small k -> excitation relaxes towards small k -> inverse cascade.
- This approach only depends on the ideal quadratic conserved quantities of the system, so we can then obtain an indication of the cascade directions in 2D CHNS by changing the name in variables.

Physics System	Conserved Quantity	Cascade Direction
2D MHD	E_k	Direct
	H_k^A	Inverse
2D CHNS	E_k	Direct
	H_k^ψ	Inverse
2D NS	E_k^K	Inverse
	Ω_k	Direct

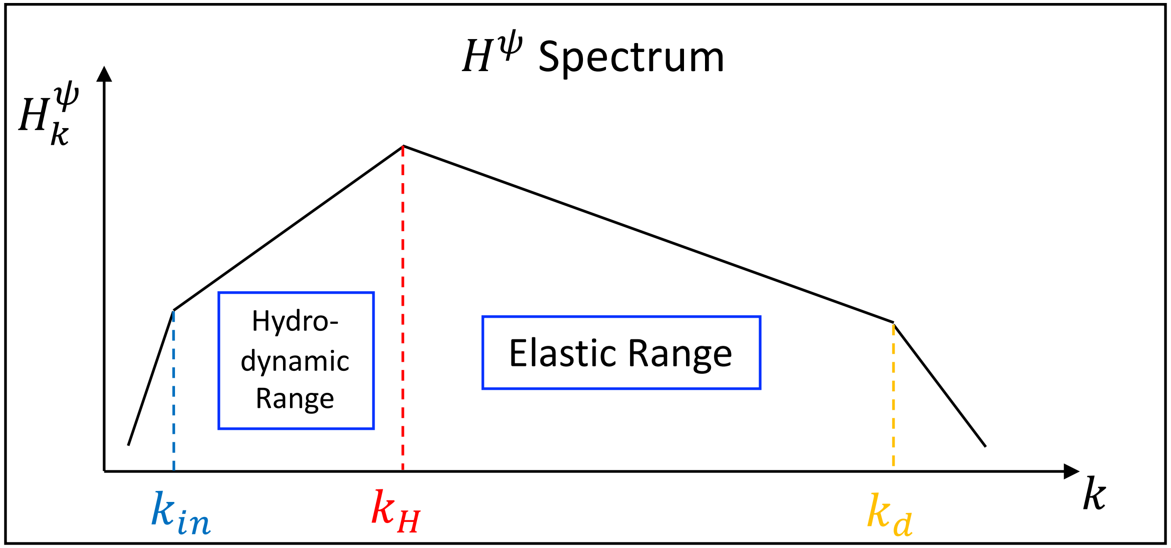
Important length scales and ranges

- The statistically stable blob size is modelled by the Hinze scale.
- **Hinze scale**: the balance between blob merger and blob breakup processes, i.e. between turbulent kinetic energy and surface tension energy.
- In the 2D NS direct enstrophy cascade regime, the velocity distribution is $\frac{\langle v^2 \rangle}{k_H} \sim \epsilon_\Omega^{2/3} k_H^{-3}$ (It will be explained later why -3). So

$$L_H \sim (\frac{\rho}{\xi})^{-1/3} \epsilon_\Omega^{-2/9}$$

- We define the scales between L_H and dissipation scale L_d to be **the elastic range**, where the blob coalescence process dominates.
- Define a dimensionless number: $\frac{L_H}{L_d} = Hd \sim (\frac{\rho}{\xi})^{-1/3} \nu^{-1/2} \epsilon_\Omega^{-1/18}$
- $\frac{L_H}{L_d} = Hd \gg 1$ is required to form a long enough elastic range.

Important length scales and ranges



- In the elastic range of the 2D CHNS system, the blob coalescence process is analogous to the magnetic flux coalescence process in 2D MHD.
- The former leads to the inverse cascade of H^{ψ}, and the latter leads to the inverse cascade of H^{A} .
- In the elastic range of the 2D CHNS system, surface tension induces elasticity and plays a major role in defining a restoring force. Similarly, in 2D MHD, the magnetic field induces elasticity and make MHD different from a pure NS fluid.
- The 2D CHNS system is more MHD-like in the elastic range.

UC San Diego

Simulation Setup

- The PIXIE2D code [Chacon 2002, 2003] is used to simulate the system. PIXIE2D originally solves the 2D MHD equation, and now is modified to be able to solve the Cahn-Hilliard Navier-Stokes (CHNS) equations, too. It is a Direct Numerical Simulation (DNS) code that solves the following equations in real space:

$$\partial_t \psi + \vec{v} \cdot \nabla \psi = D \nabla^2 (-\psi + \psi^3 - \xi^2 \nabla^2 \psi) + F_\psi$$

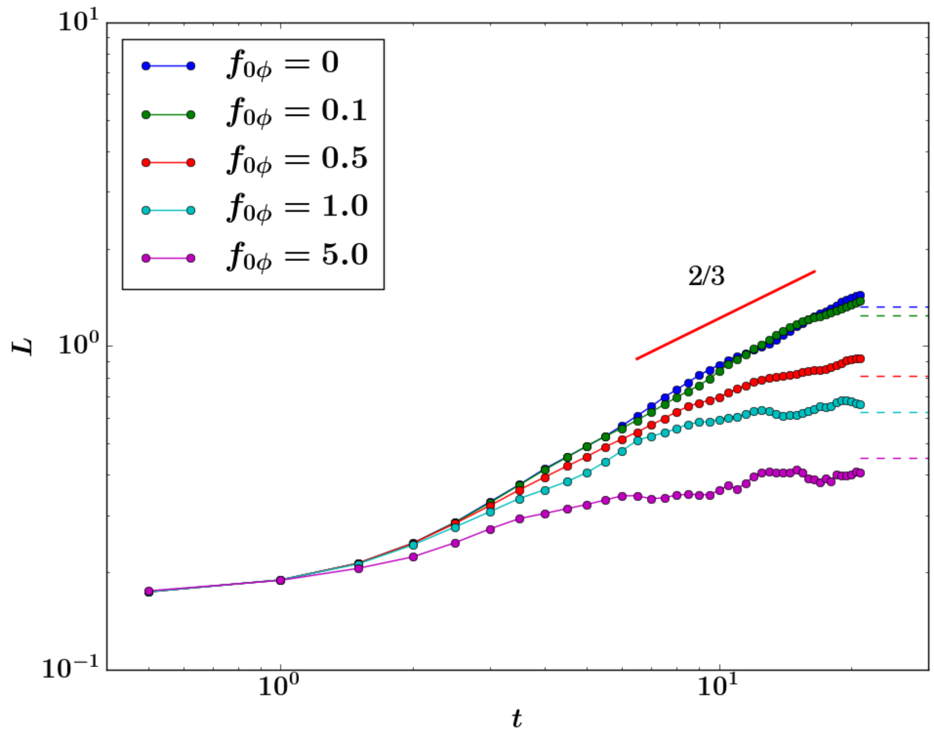
$$\partial_t \omega + \vec{v} \cdot \nabla \omega = \frac{\xi^2}{\rho} \vec{B}_\psi \cdot \nabla \nabla^2 \psi + \nu \nabla^2 \omega + F_\omega$$

With $\vec{v}=\hat{\hat{z}} \times \nabla \phi$, $\omega = \nabla^2 \phi$, $\vec{B}_\psi = \hat{\hat{z}} \times \nabla \psi$, $j_\psi = \xi^2 \nabla^2 \psi$

- Initial condition: ψ in each cell is assigned to 1 or -1 randomly; $\phi = 0$ everywhere.
- Boundary condition: doubly periodic.
- External force for A and ϕ: an isotropic homogeneous force that has a wave number k_{in}:

$$f_{A,\psi,\phi}(x,y) = f_{0A,\psi,\phi} \sin[x * \mathrm{int}(k_{fA,\psi,\phi} \cos\theta_{A,\psi,\phi}) + y * \mathrm{int}(k_{fA,\psi,\phi} \sin\theta_{A,\psi,\phi}) + \varphi_{A,\psi,\phi}]$$

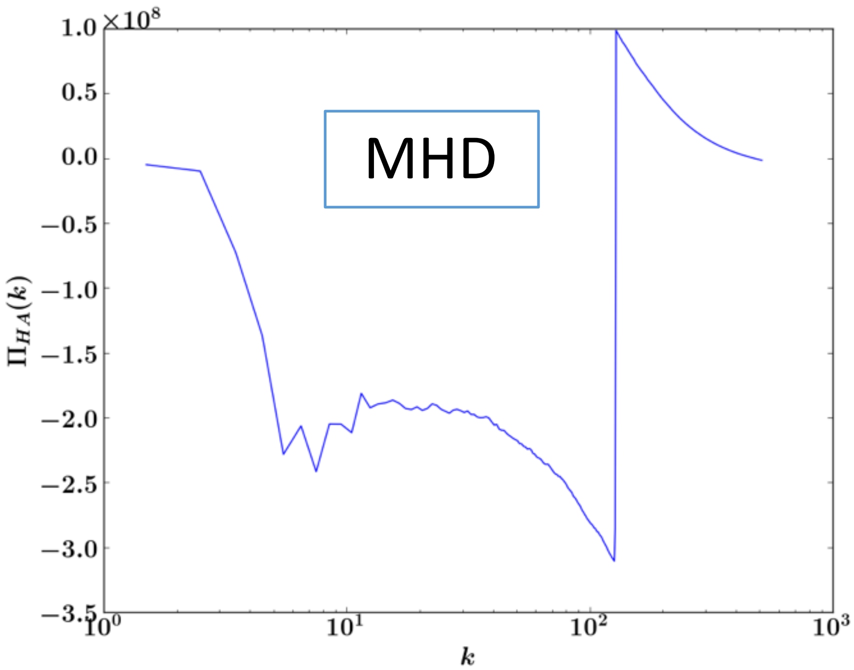
Benchmark

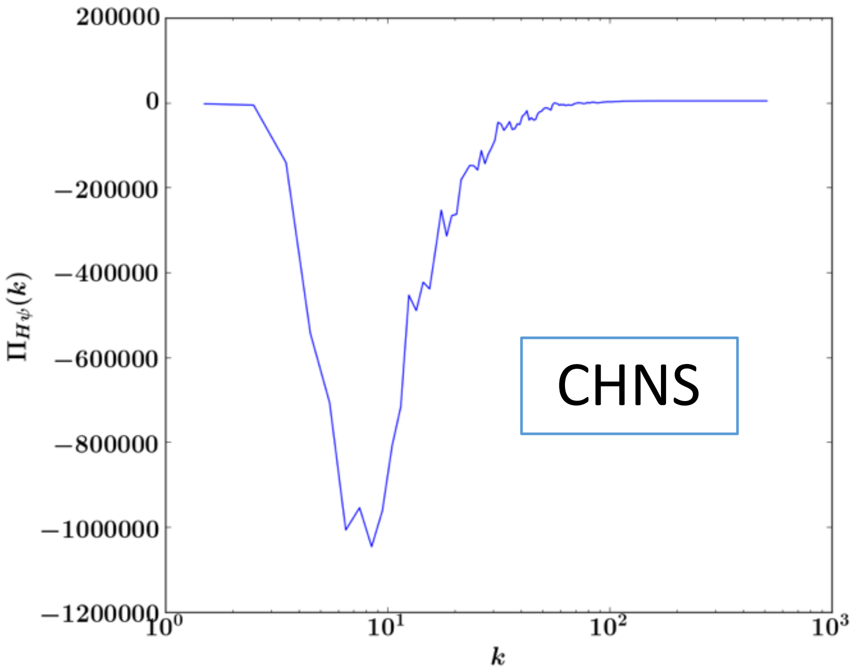


- We verified the length scale growth is $L(t) \sim t^{2/3}$; and the growth can be arrested. The saturation length scale is consistent with the Hinze scale (dashed line).

Spectral Fluxes

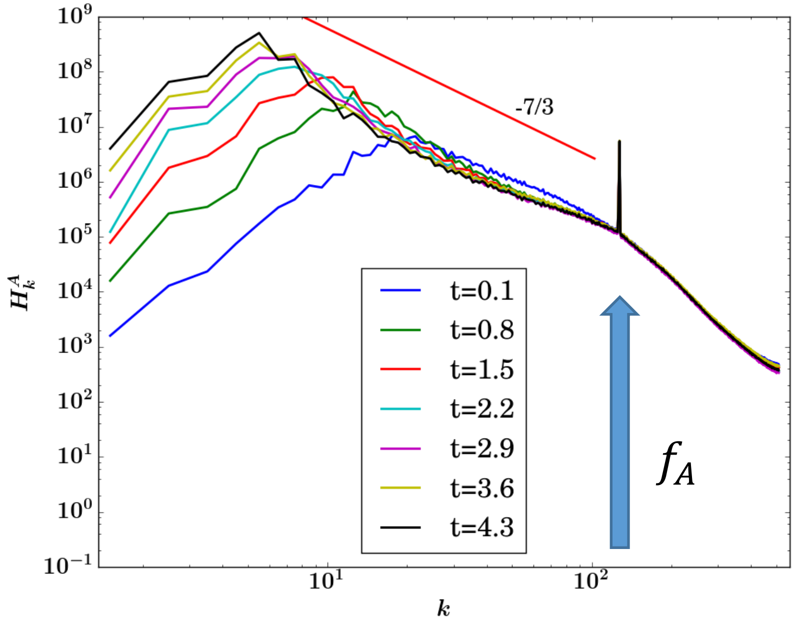
- The spectral fluxes are negative, and this indicates inverse cascade of H^A and H^ψ.
- For the MHD case (left), an external forcing on the magnetic potential A is applied on k = 128. The small scale A forcing drives an inverse transfer of H^A. For the CHNS case (right), no forcing on ψ is necessary for the appearance of an inverse transfer of H^ψ.





H_k^A / H_k^ψ spectrum power law: -7/3

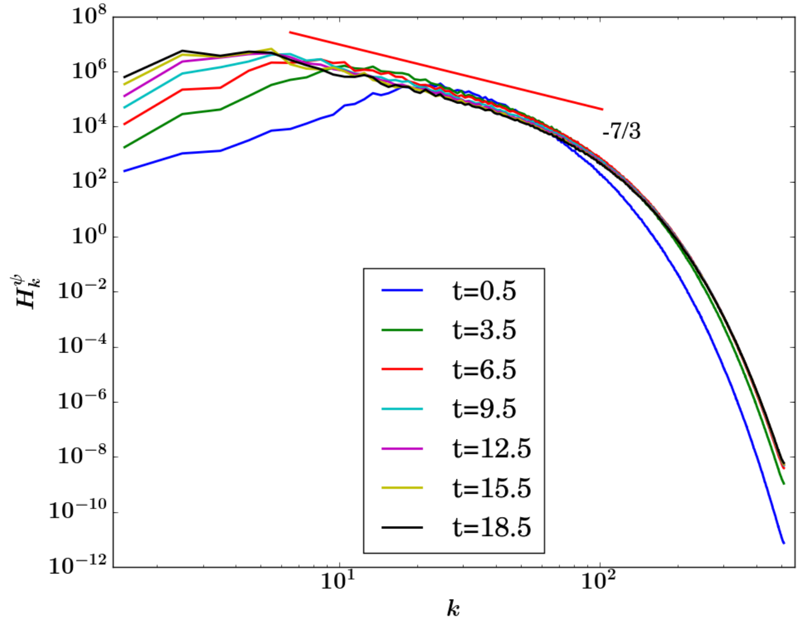
MHD



$$H_k^A \sim \epsilon_{HA}^{2/3} k^{-7/3}$$

Inverse cascade of H^A

CHNS

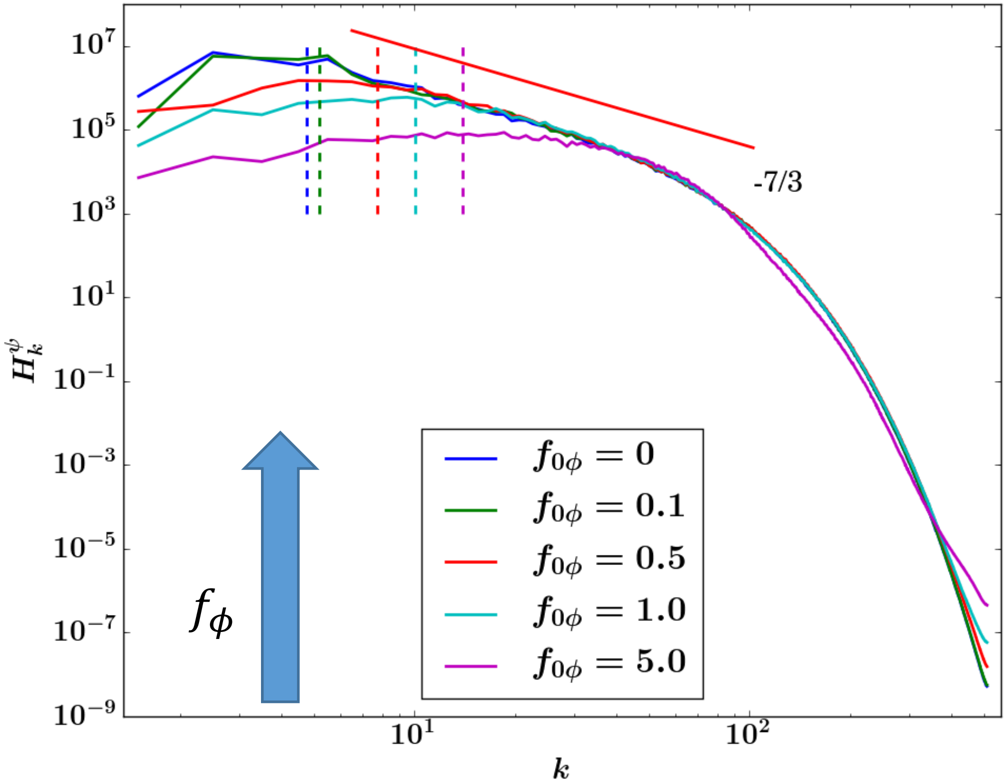


$$H_k^\psi \sim \epsilon_{H\psi}^{2/3} k^{-7/3}$$

Inverse cascade of H^ψ

H_k^{ψ} / H_k^{A} spectrum power law: -7/3

- The −7/3 power is robust. It does not change with the magnitude of external forcing, as long as the separation between the Hinze scale and the dissipation scale is maintained, so the elastic range is long enough.



H_k^A/H_k^ψ spectrum power law: -7/3

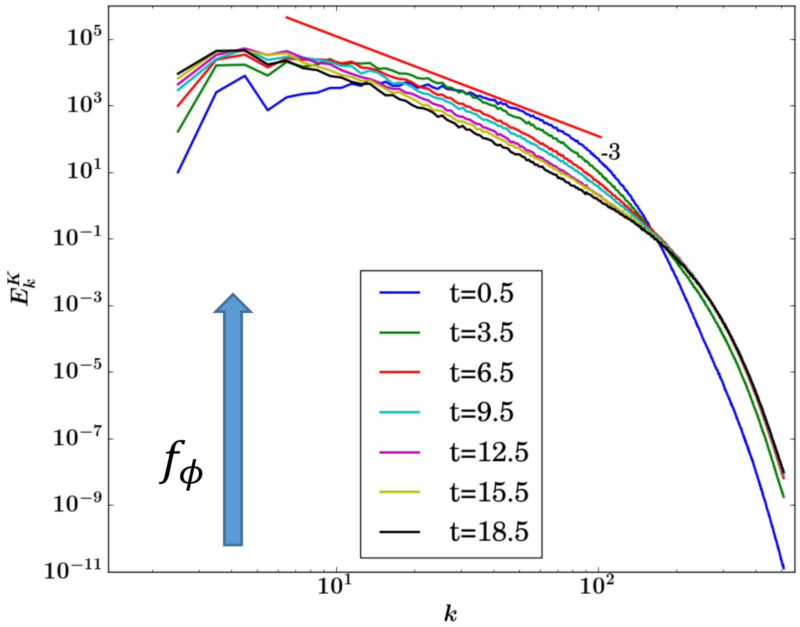
- Assuming a constant mean square magnetic potential dissipation rate ϵ_{HA}, according to the Alfvenic equipartition ($\rho\langle v^2\rangle \sim \frac{1}{\mu_0}\langle B^2\rangle$), the time scale for the decay of H^A ($\epsilon_{HA}\sim H^A/\tau$) can be estimated by $\tau\sim(vk)^{-1}\sim(Bk)^{-1}$.
- Define the spectrum to be $H^A=\sum_k H_k^A\sim kH_k^A$, so $B\sim kA\sim k(H^A)^{\frac{1}{2}}\sim(H_k^A)^{\frac{1}{2}}k^{\frac{3}{2}}$. Therefore $\epsilon_{HA}\sim\frac{H^A}{\tau}\sim(H_k^A)^{\frac{2}{3}}k^{\frac{7}{2}}$.

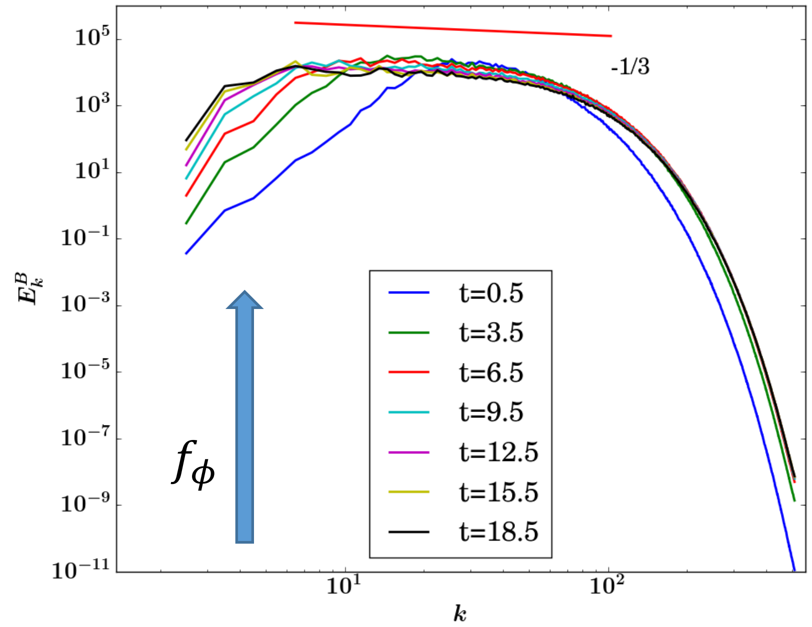
$$\boxed{H_k^A\sim\epsilon_{HA}^{2/3}k^{-7/3}}$$

- Similarly, we can obtain the H_k^ψ spectrum by assuming the elastic equipartition ($\rho\langle v^2\rangle \sim \xi^2\langle B_\psi^2\rangle$):

$$\boxed{H_k^\psi\sim\epsilon_{H\psi}^{2/3}k^{-7/3}}$$

Energy Spectrum Power Law





- $E_k^K \sim k^{-3}$ in 2D CHNS turbulence; on the other hand, it is well known that $E_k^K \sim k^{-3/2}$ in MHD turbulence.
- The -3 power law is consistent with the direct enstrophy cascade in 2D NS turbulence.
- The -3/2 power law comes from the Alfven effect (Iroshnikov-Kraichnan Theory).

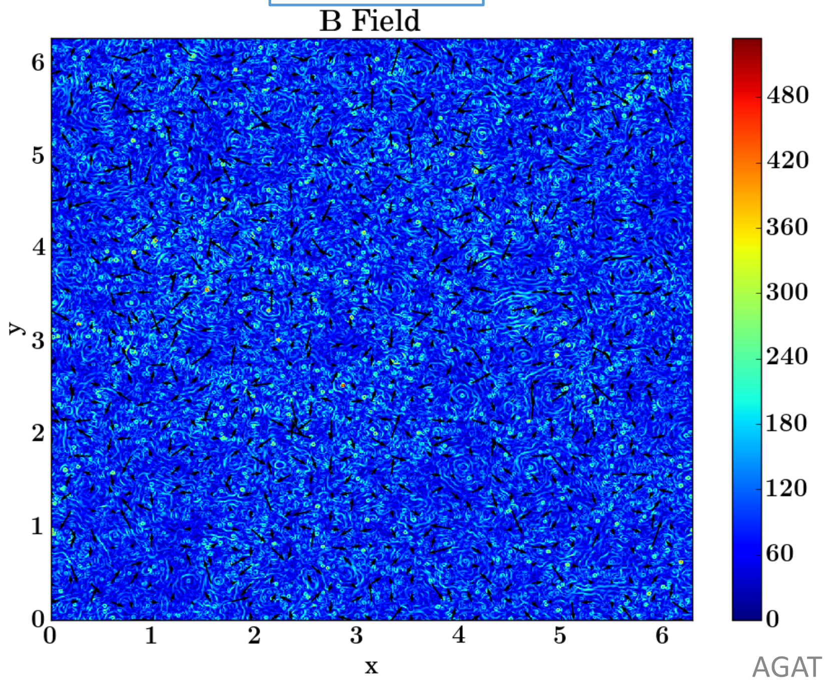
Energy Spectrum Power Law

- $H_k^{\psi} \sim k^{-7/3}$ vs. $H_k^{A} \sim k^{-7/3}$, the powers are the same.
- $E_k^{K} \sim k^{-3}$ for 2D CHNS; $E_k^{K} \sim k^{-3/2}$ for 2D MHD. Why? Why not -3/2?

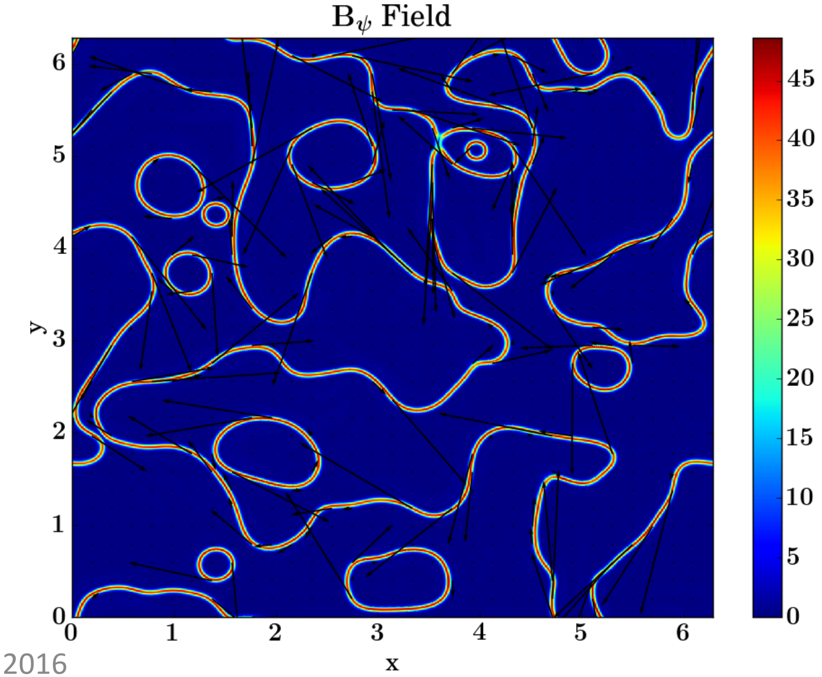
Energy Spectrum Power Law

- This initially surprising result is plausible because in the 2D CHNS system, B_ψ vanishes in most regions. Back reaction is apparently limited.
- On the other hand, the magnetic fields in MHD are not localized at specific regions, and Alfven waves can be everywhere.

MHD



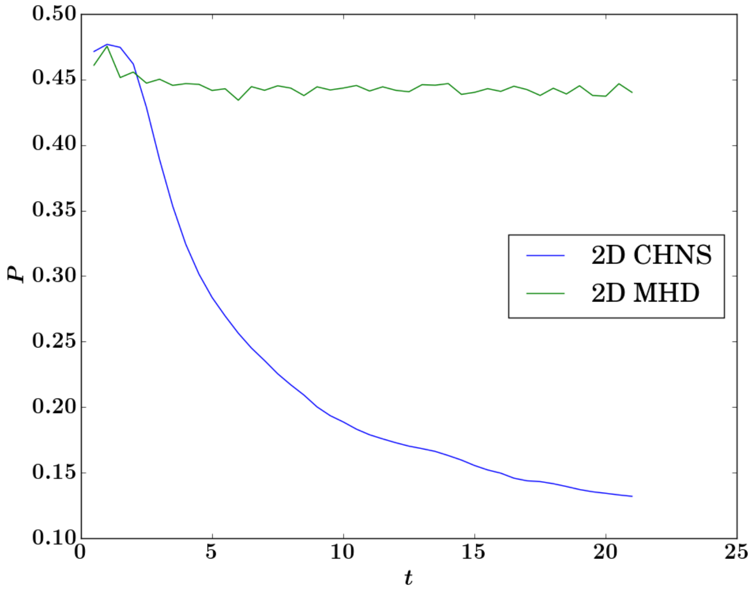
CHNS



Energy Spectrum Power Law

- Define the interface packing fraction P to be the ratio of mesh grid number where $|\vec{B}_\psi| > B_\psi^{rms}$ (or $|\vec{B}| > B^{rms}$) to the total mesh grid number.

$$\partial_t \omega + \vec{v} \cdot \nabla \omega = \frac{\xi^2}{\rho} \vec{B}_\psi \cdot \nabla \nabla^2 \psi + \nu \nabla^2 \omega$$



Summary

- Comparison:

	2D MHD	2D CHNS
Ideal Quadratic Conserved Quantities	Conservation of E, H^A and H^C	Conservation of E, H^ψ and H^C
Role of elastic waves	Alfven wave couples $\mathbf{v}$ with $\mathbf{B}$	CHNS linear elastic wave couples $\mathbf{v}$ with $\mathbf{B}_\psi$
Origin of elasticity	Magnetic field induces elasticity	Surface tension induces elasticity
Origin of the inverse cascades	The coalescence of magnetic flux blobs	The coalescence of blobs of the same species
The inverse cascades	Inverse cascade of H^A	Inverse cascade of H^ψ
Power law of spectra	$H_k^A \sim k^{-7/3}$	$H_k^\psi \sim k^{-7/3}$

- Contrast:

	2D MHD	2D CHNS
Diffusion	A simple positive diffusion term	A negative, a self nonlinear, and a hyper-diffusion term
Range of potential	No restriction for range of A	$\psi \in [-1, 1]$
Kinetic energy spectrum	$E_k^K \sim k^{-3/2}$	$E_k^K \sim k^{-3}$
Suggestive cascade by E_k^K	Suggestive of direct energy cascade	Suggestive of direct enstrophy cascade
Back reaction	$\mathbf{j} \times \mathbf{B}$ force can be significant	Back reaction is apparently limited